Determination of Limit Bearing Capacity of Statically Indeterminate Truss Girders

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***Abstract* – After the theoretical base of the structure limit analysis being presented, this paper presents the algorithm for calculation of limit load in statically indeterminate truss girders. The algorithm is based on the application of matrix structure analysis and is used for determining static and deformation values within the structural limit analysis step by step method, which gives a detailed view into the behavior of a girder in a gradual load increasing, until the formation of failure mechanism. This algorithm has been the base for making a computer program for determining the load limit in truss girders and this has been presented in this paper in numerical example and also presented in this example is the application of linear programming in determination of the limit load.**

***Keywords* – Step by step method, limit load, linear programming, truss girders.**

1. Introduction

Theory of plasticity is a part of mechanics which deals with stress calculation and deformation of a body, made of ductile material, permanently deformed by the acting of an external load. In contrast to elastic bodies, where deformation depends only on the final state of stress, the determination of deformation that appears in a plastic body, requires analysis of the whole history of load acting. Problem of plasticity is, therefore, mainly incremental, so that final displacements of bodies are defined as a total sum of incremental displacements.

The subject of this work are statically indeterminate truss girders, exposed to the action of load that proportionally increases and leads to the formation of failure mechanism which totally exhausts bearing capacity of the structure.

In order to determine a limit bearing capacity of a structure by applying theory of plasticity, it is necessary previously to prove that its limit state will appear by formation of failure mechanism, i.e. it is needed to eliminate the appearance of any other limit states. It is necessary to exclude the occurrence of fatigue caused by an variable load, as well as the possibility of the appearance of the local instability before reaching a full plasticity, and also to exclude the occurrence of any effects which would lead to structural failure before enough number of plastic bars are formed in truss girders, for its transition to the failure mechanism.

Many materials (e.g. most metals) show plastic behavior, that is, they are ductile. Even when the stress value reaches the yield stress value, ductile materials can considerably deform, without break, which proves the fact that although the stress intensity at a definite point of statically indeterminate structure reaches critical value (yield stress), the construction need not break or considerably deform. Instead, there is a redistribution of stress so that it is possible a certain increasing of load. Structural failure occurs only after the failure mechanism, i.e. after the occurrence of a constant plastic yielding. Thus, a real bearing capacity of a structure is higher (in some cases in a considerable extent) than that calculated by an elastic analysis which prove many experiment results, too.

Although some new ideas appeared in the 18th century, limit analysis is of more recent date. Its origins are linked to Kazincy (1914), who calculated failure load in fully fixed beam and confirmed it experimentally. A similar concept was proposed by both Kist (1917) and Gruning (1926). But the early works from this field were mainly based on engineering intuition. Although the static theorem was first proposed by Kist (1917), as an intuitive axiom, it is still considered that the basic limit analysis theorems were first presented by Gvozdev in 1936. The possibility of more economic structural designing based on the structural limit analysis has been attracting the attention of design engineers for years and the first works with some simple constructions appeared 60 years ago. Beside the possibility of accepting higher loading, the limit analysis proved to be much easier for work than the classical elastic analysis.

2. Basic settings of the structural limit analysis

Diagram of dependence between stress (σ) and deformation (ε) for mild steel (in further text σ-ε diagram), the metal that is most used in civil engineering, is shown on Figure 1(a). For the needs of structural analysis, σ-ε diagrams are idealized as it is shown in Figure 1(b), which presents an ideal elastic-plastic behavior.

Stress σ cannot exceed limits of the yield stress at tension σ_t^+ and the yield stress at compression σ_t^-. With metals, both two yield stresses (at compression and tension) have the same values $\sigma_t^+ = \sigma_t^- = \sigma_t$.

The diagram which corresponds to the ideally elastic-plastic material, Figure 1(b), that is most applied to the calculation according to the limit analysis, does not take into account hardening of materials. However, researches show that neglecting of the material hardening phenomenon does not make any significant errors, especially if the material has a large yielding area, because with such materials, hardening occurs only when the section bearing capacity is completely exhausted [1].

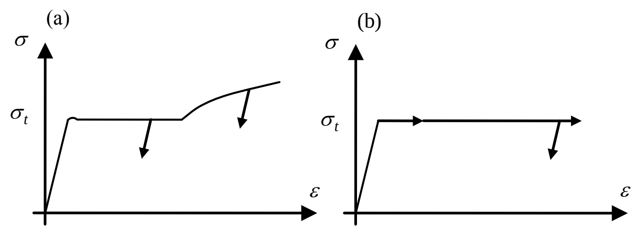

Figure 1. (a) $\sigma - \varepsilon$ diagram for mild steel; (b) Model of an ideal elastic-plastic behavior

When the stress in the bar reaches the value of the yield stress σ_t the bar begins to "yield" and the stress cannot any longer increase. Thus, the force in the bar remains constant and equal to the force of full plasticity $S_p=A\sigma_t$. Similar observation can be taken in the case of a bar exposed to the force of compression. The bar which has become fully plasticized and which cannot accept any more load increasing is called ***plastic bar***.

On the basis of recommendations in EUROCODE, the allowed stress in compressed bars (σ_{ti}) can be calculated by the reduction of the yield stress according to the expression (1), and the force of full plasticity of the compressed bars by the equation (3), [2]:

$$\sigma_{ti} = \varphi\sigma_t \ , \qquad (1)$$

$$\varphi = \beta - \sqrt{\beta^2 - \frac{b^2}{\lambda^2}\beta} = \frac{1 + a(\frac{\lambda}{b} - 0.2) + (\frac{\lambda}{b})^2}{2(\frac{\lambda}{b})^2}, \qquad (2)$$

$$S_p^- = A\sigma_{ti} . \qquad (3)$$

Here, λ is slenderness of a bar and it is calculated in the common way. Constants *a* and *b* depend on the shape of the bar cross-section and on the material qualities the bar is made of.

In the initial phase of load, the structure behaves elastic so as the standard procedures of the linear elastic analysis can be applied. The corresponding solution is valid unless the plastic yielding occurs in a certain point of construction. However, the bearing capacity of a structure usually is not immediately exhausted, so the load can be more increased which increases plastic area, too.

In truss girders, the stress in each bar is constant, so the yielding starts simultaneously at all the points of the bar. Strictly speaking, this is true only in the ideal case of a bar with the ideal cross-section, made of the ideally homogenous material, etc. Because of the unavoidable geometric and material irregularities, yielding starts at the most critical point, but the plastic area soon spreads to at least one cross-section, so the assumption of constant force in the bar after the beginning of yielding is justified [3]. Axial force which is transmitted by the bar, is constant, so the additional load increase must be accepted by other bars which are still in the elastic area. This means that the structure, under further load increase, behaves as if the plastic bar did not exist. If the structure is statically determined, it changes into kinematic mechanism as soon as the yielding starts in the first bar, so the failure occurs on the elastic limit. However, if the degree of the statically indeterminate structure is $r \geq 1$, then after the beginning of yielding in the first bar, the degree of indeterminateness becomes $r-1$, so the structure can accept more load increase. When the stress in the second bar reaches the yield stress, the degree of the static indeterminateness becomes $r-2$, etc.

In statically indeterminate truss girders, failure mechanism occurs when the number of plastic bars is higher than the number of static indeterminateness. Magnifying the number of plastic bars, failure mechanism with one degree of freedom occurs. Failure mechanisms with two degrees of freedom cannot occur having in mind that after formation of failure mechanism with one degree of freedom occurs before the plasticization of the following bar. In fact, failure can occur even if there are less than r+1 of plastic bars, in the case of the partial failure mechanism formation. Then, a part of the truss remains statically indeterminate, but one or more joints can move without extension of the bars which are still in the elastic area.

During every phase among the formation of plastic bars, incremental behavior corresponds to the elastic construction from which bars being plasticized have been removed and their influence replaced by constant axial forces that are equal to forces of full plasticity of the removed bars. This fact can be used for formation the algorithm of elastic-plastic analysis and it will be observed further in this paper.

3. Matrix formulation of the problem of truss girders limit analysis

Analysis of truss girders, applying matrix analysis, is a method suitable for the analysis of truss girders using a computer, particularly with structures with a large number of joints and bars. The aim of this analysis is determining of forces in bars as well as the displacement of joints of the truss girder under the action of external load that acts in the truss joints and it covers three elementary groups of equations:

a) Equilibrium conditions: $[B]\{s\}=\{f\}$; (4)

b) Relationships of internal forces in truss bars and the extensions of bars: $\{s\}=[k]\{\delta\}$; (5)

c) Conditions of deformation compatibility and displacement: $\{\delta\}=[B]^T\{u\}$, (6)

where the:

$\{s\}$ - vector of order *n* whose elements of force are in bars; *n* – number of bars,

$[B]$ - static or equilibrium matrix, (values $n\times m$), which gives relationship between unknown forces in bars and the external load; m – number of unknown displacements,

$\{f\}$ - vector of order *m* whose force elements are in the joints of the truss girder,

$[k]$ - square matrix (material matrix of the girder stiffness) of order *n* whose diagonal elements are the individual stiffness of truss bars, whereas outdiagonal elements are equal to zero,

$\{\delta\}$ - vector of order *n* whose elements are the extensions of bars,

$[B]^T$ - compatibility matrix (kinematic matrix) gives the link between the bar deformation and the girder joints displacement; it is obtained by transposition of static matrix,

$\{u\}$ - vector of order *m*, whose elements are unknown displacements of girder joints.

If in the equations, the force balances in bars are shown by the expression given while deriving constitutional equations and then the extensions of bars are expressed in the way given within the compatibility equations, we obtain the equation system in which the unknowns are unknown displacements of the girder bars:

$$[K]\{u\}=\{f\}, \quad (7)$$

where the:

$$[K]=[B][k][B]^T, \quad (8)$$

is the stiffness matrix of the girder, which in complex structures is not determined in this way (because it requires multiplication of big matrixes), but by forming the girder rigidity matrix from the stiffness matrixes of the individual truss bars.

In the expression (7) the only unknowns are the joint displacements of the truss girder after formation of the stiffness matrix, and with the known load vector, they are easy to be determined:

$$\{u\}=[K]^{-1}\{f\}. \quad (9)$$

Expression for the forces in the truss bars, in the function of the known displacements, is obtained by applying the compatibility and constitutive equation:

$$\{s\}=[k]\{\delta\},$$

$$\{s\}=[k][B]^T\{u\}. \quad (10)$$

The girder is first in the elastic area and the values of the forces in bars are determined by the expression (10). When the force in the most loaded bar reaches the value of the full plasticity force, plastic bar is formed, and then it is accepted that the bar stiffness in the material matrix of stiffness is equal to zero, which changes the girder matrix of stiffness as well [4]. Thus, under load increase, plastic bar cannot accept the load, i.e. the force in that bar cannot increase. At statically indeterminate girders, after formation of each plastic bar, change of material matrix of stiffness appears as well as the stiffness matrix of the girder, and at the same time, the change of the expression for joint displacement and the force in the truss bars, too. After formation of the r+1-the plastic bar, stiffness matrix of the girder is singular (det K = 0) which points to the fact that failure mechanism has been formed. In the case of partial failure mechanism, stiffness matrix is not singular.

4. Algorithm for determination of limit load

Previously shown algorithm for determination of limit load can be shown in the following steps:

1. On the basis of geometric characteristics of a girder (position of girder joints relative to global coordinate system, cross-sectional areas of bars, material modulus of elasticity), static matrix (*B*) and current material matrix of stiffness (*k*) are calculated, and based on data load, load vector (*f*) is defined, too.

2. Using the expressions (8), (9) and (10) we determine stiffness matrix (*K*), joint displacements (*u*) and the force in bars (*s*) in the function of load parameter (μ).

3. Equalizing the force values in bars with the full plasticity forces ($S_{p,i}$), we obtain the load at which plastification of the first bar occurs (lowest value). Using that load value, we obtain force values in bars at the moment when the girder transits from elastic to elastic-plastic state.

4. In material matrix of stiffness, we accept that stiffness in the bar that has been plasticized is equal to zero and then using (8), new stiffness matrix is obtained.

5. The loop at which steps *a*-*e* are repeated until one of these conditions are met:

- stiffness matrix is singular or
- partial mechanism has been formed

a) Using new stiffness matrix, we determine the values of forces in bars in the function of load growth ($\Delta\mu$)
b) Equalizing the necessary force growth in the bar (difference between the bar full force of plasticity and the bar force at the moment of formation of the previous plastic bar) with the bar force in the function of load growth, necessary load growth for each bar is obtained. In the bar in which the lowest growth is needed, plastification, i.e. formation of a new plastic bar, will occur.
c) Using the obtained value of load growth, forces in the bar and the displacement of girder joints are calculated.
d) It is adopted that bar stiffness, in which plastificaton has occurred, is equal to zero, and a determinant of stiffness matrix is calculated.
e) If stiffness matrix is not singular and partial mechanism is not formed, the whole procedure goes back to the beginning of the loop. If one of these two conditions is met, loop ends and the value of limit load is obtained.

Using the shown algorithm, there has been made the program in the program package MATLAB, which is shown in [4]. Applying this program, the value of limit load for girders in the Example is determined.

5. Determination of limit bearing capacity using linear programming

The solutions based on the methods determining the maximum statically possible parameter or minimal kinematically possible parameters have not been systematized, and are partially based upon the engineers' intuition. Such approach is not adequate in design of large, real structures. Fortunately, it proved that the problem of limit analysis can also be formulated as a problem of linear programming, so the methods developed in the mathematical optimization theory can be applied in the limit structural analysis [4].

According to the theorem of the lower limit of failure load, the safe parameter μ can be determined as the highest possible statically possible increase parameter. The statically possible state is characterized by the increase parameter μ_s and the internal forces vector *s*, which meet the equilibrium equations and the plasticity condition. The corresponding problem of linear programming can be expressed as:

$$\max \quad f(s,\mu_s) \equiv \mu_s , \qquad (11) \qquad (7)$$

$$B\,s = \mu_s f , \qquad (12) \qquad (8)$$

$$-s_p \le s \le s_p , \qquad (13)$$

where:

s_p – vector of full plasticity of members forces,

6. Example

The force of full plasticity of the bars S_1-S_5 is $2S_p$, and the force of full plasticity of the bars S_6-S_{10} is S_p. It is assumed that the force of full plasticity with compression and tension is the same.

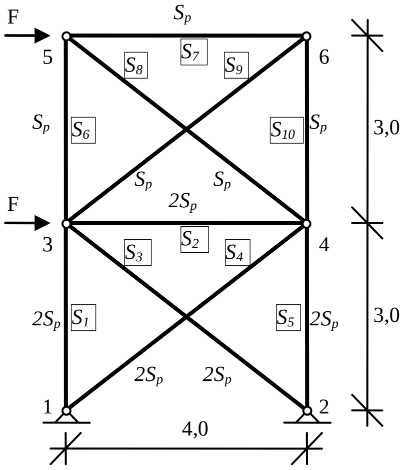

Figure 2. Truss girder loaded by horizontal load

By entering the modulus of elasticity and yield stress of the material, the girder has been made of, the position of joints and the characteristics of bars, the program calculates static matrix, material stiffness matrix and the stiffness matrix of the girder. After the load being defined by the user, the program gives the force vector in the bars and the vector of girders joints displacement and then determines under what loading the girder transits to elastic-plastic state (*1,2901* S_p) and what values of the forces in bars are at the moment. The obtained values of the forces in bars are shown in the second column of Table 1.

At further load growth, the bar S_1 in which plastification has occurred, cannot receive loading, i.e. the girder behaves as if that bar did not exist, that is why it is adopted that the bar stiffness is zero (*k(1,1)=0*). The program automatically calculates a new stiffness matrix of the girder, bar forces in the function of load growth, the required load growth for the plastification of each truss bar and adopts the minimal growth as a competent. With that growth, it calculates the growth required for the plastificaton of the second bar (*1,2991* S_p) and the force values in bars at that load (third column in Table 1.).

	loading		
bar	*1,2901 S_p*	*1,2991 S_p*	*1,3333 S_p*
S_1	***2 S_p***	***2 S_p***	***2 S_p***
S_2	*-0,4107 S_p*	*-0,4308 S_p*	*-0,5333 S_p*
S_3	*-1,7207 S_p*	*-1,7094 S_p*	*-1,6667 S_p*
S_4	*1,5046 S_p*	*1,5385 S_p*	*1,6667 S_p*
S_5	*-1,8703 S_p*	*-1,8974 S_p*	***-2 S_p***
S_6	*0,5947 S_p*	*0,60 S_p*	*0,60 S_p*
S_7	*-0,4972 S_p*	*-0,4991 S_p*	*-0,5333 S_p*
S_8	*-0,9912 S_p*	***- S_p***	***-S_p***
S_9	*0,6214 S_p*	*0,6239 S_p*	*0,6667 S_p*
S_{10}	*-0,3729 S_p*	*-0,3744 S_p*	*-0,40 S_p*

Table 1. Forces in bars at gradual load increasing

For further load growth, the program adopts that the stiffness of the bar S_8, in which plasticity has occurred, is equal to zero (*k(8,8)=0*), determines a new stiffness matrix and checks whether it is singular, which would mean that a global failure mechanism has been formed. Since this is not the case, the user has to define whether failure mechanism has been formed because the program cannot recognize it. As neither of these two conditions are not met, the program continues to calculate. The fifth step from the algorithm is repeated, i.e. the required growth for the plastification of the following bar is determined, minimum value is adopted and the load, at which the plastification will occur, is determined (*1,3333 S_p*) and the force values in bars at that load (fourth column of Table 1). We adopt that the stiffness of the bar S_5, in which plastification has been formed, is equal to zero (*k(5,5)=0*) and check whether the stiffness matrix of the girder is singular. As this condition is met, the program ends and the obtained load is declared a limit load.

Transformation of the girder, i.e. its behavior in a gradual increase in load, can be clearly noted on Figure 3.

The linear programming problem for girder shown in Figure 2. is given by the expressions (11)-(13) which, for the given girder have the following form:

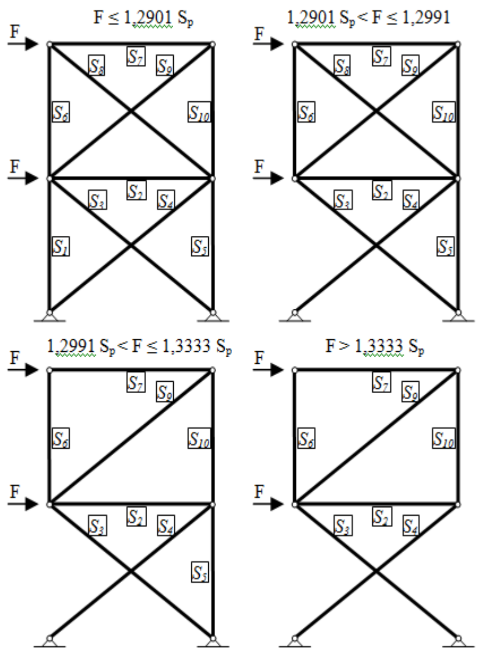

Figure 3. Behavior of girder in a gradual increase in load

$$\max f(s,\mu) \equiv \mu \Rightarrow \min \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \\ -\mu_s \end{bmatrix} = \min \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{14}$$

$$\begin{bmatrix} 0 & -1 & -0,80 & 0 & 0 & 0 & 0 & 0 & -0,80 & 0 & -1 \\ 1 & 0 & 0,60 & 0 & 0 & -1 & 0 & 0 & -0,60 & 0 & 0 \\ 0 & 1 & 0 & 0,80 & 0 & 0 & 0 & 0,80 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0,60 & 1 & 0 & 0 & -0,60 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -0,80 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0,60 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0,80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,60 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \\ \mu_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} -2S_P \\ -2S_P \\ -2S_P \\ -2S_P \\ -2S_P \\ -S_P \\ -S_P \\ -S_P \\ -S_P \\ -S_P \\ 0 \end{bmatrix} \le \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ \mu \end{bmatrix} \le \begin{bmatrix} 2S_P \\ 2S_P \\ 2S_P \\ 2S_P \\ 2S_P \\ S_P \\ S_P \\ S_P \\ S_P \\ S_P \\ 10S_P \end{bmatrix}, \qquad (16)$$

Using the software package MATLAB the value of the function $f(s,\mu) \equiv \mu$, is obtained, that is the value of the forces in members in the moment of the limit equilibrium, as well as the value of the failure load parameter μ (17).

$$f(s,\mu) = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \\ \mu_s \end{bmatrix} = \begin{bmatrix} 2 \\ -0{,}5333 \\ -1{,}6667 \\ 1{,}6667 \\ -2 \\ 0{,}60 \\ -0{,}5333 \\ -1 \\ 0{,}6667 \\ -0{,}40 \\ 1{,}3333 \end{bmatrix} S_P . \qquad (17)$$

7. Conclusion

The paper presents the determination of limit load for statically indeterminate truss girders using the step by step method (incremental analysis).

The paper indicates that instead of the classical way of calculating the impact in the incremental analysis, the calculation can be accelerated by the application of the matrix structure analysis, particularly because that way of problem formulating is suitable for forming a computer program based on the algorithm which is given in this paper. This problem is reduced to defining the basic characteristics of the girder and the load, with the necessary follow-up program execution by the user which is primarily reflected in the recognition of a possible formation of partial mechanism. This concept of using incremental analysis eliminates its basic lack which is the time required for the calculation of limit load. The application of such a program requires only a few minutes for the calculation of the failure load in girders, given in the Example, which greatly points to the necessity of using this approach in the limit analysis.

Apart from the limit analysis methods based on incremental analysis, also presented is the application of linear programming in determination of the limit load as one of the fundamental methods of contemporary structural limit analysis.

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